Magnetic properties of SO(5) superconductivity

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Abstract

The distinction between type I and type II superconductivity is re-examined in the context of the SO(5) model recently put forth by Zhang. Whereas in conventional superconductivity only one parameter (the Ginzburg-Landau parameter κ) characterizes the model, in the SO(5) model there are two essential parameters. These can be chosen to be κ and another parameter, β, related to the doping. There is a more complicated relation between κ and the behavior of a superconductor in a magnetic field. In particular, one can find type I superconductivity, even when κ is large, for appropriate values of β.

1   Introduction

Several years ago, the SO(5) model was proposed as a description of high-temperature superconductors [1]. At low temperatures these materials exhibit superconductivity (SC) or antiferromagnetism (AF) depending on the doping; the SO(5) model attempts to unify these two phenomena (both of which involve spontaneous symmetry breaking) into a single symmetry group.

Not surprisingly, the SO(5) model is somewhat more complicated than the corresponding model describing conventional superconductivity (which is based on the smaller group SO(2)); correspondingly, richer behavior can be seen. In conventional superconductivity, one parameter, the Ginzburg-Landau (GL) parameter κ, determines the behavior. For instance, $\kappa = 1/\sqrt{2} \equiv \kappa_c$ marks the boundary between type I and type II superconductors.

In the SO(5) model, there are two dimensionless parameters, which can be taken to be κ and a second parameter, β (to be defined below); the latter is related to the degree to which SO(5) is explicitly broken (by doping, for instance). As will be shown below, this second parameter has a profound influence on the magnetic behavior of an SO(5) superconductor; for instance, the critical value of κ in the SO(5) model is given by $\kappa_c(\beta) = (1/\sqrt{2})\sqrt{(1+\beta)/(1-\beta)}$. One sees that as $\beta \to 1^-$, $\kappa_c \to \infty$, which is a dramatic departure from the conventional

value. This is of some significance since high-temperature superconductors typically have $\kappa \gg 1$, and therefore are normally thought to be extreme type II superconductors. Such a conclusion would be premature in the SO(5) model, however, since (as the above relation shows) one could have $\kappa_c > \kappa \gg 1$.

In this paper we will first remind the reader of the magnetic properties of conventional superconductors. The standard approach based on the surface energy density of a boundary between a normal phase at critical magnetic field H_c and a superconducting phase will be briefly reviewed. We will then discuss an alternative approach based on vortex energetics. Although much of the above is fairly familiar material, this discussion will establish notation and set the stage for the parallel discussion in the context of the SO(5) model.

Next, we will discuss the case of SO(5) superconductivity. After a brief review of the model itself, the critical fields H_c and H_{c2} will be calculated. From these, the critical value of the parameter κ can be calculated explicitly as a function of β. This expression can be confirmed numerically, either by examining the surface energy density or vortex energetics. In the latter approach, one can see that the dramatically different behavior in the SO(5) model is due to the possibility of an antiferromagnetic core in the vortex [2, 3, 4].

2 Review of conventional superconductivity[1]

A conventional superconductor is described by the following Helmholtz free energy in the GL theory:

$$\hat{F} = \int d\mathbf{x} \left\{ f_n - \frac{a_1^2}{2}|\phi|^2 + \frac{b^2}{4}|\phi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\nabla - \frac{e^*\mathbf{A}}{c}\right)\phi\right| + \frac{\mathbf{h}^2}{8\pi} \right\}, \quad (1)$$

where f_n is a constant, $\hat{\mathbf{h}} = \nabla \times \mathbf{A}$ is the microscopic magnetic field and a_1, b are parameters. The minimum of the potential is $|\phi|^2 = a_1^2/b \equiv v^2$.

The Helmholtz free energy is minimized when the temperature T and interior magnetic induction B are constant (i.e., thermodynamic equilibrium). One uses the Gibbs free energy for equilibrium where T and the external magnetic field H are constant. These two quantities are related by the following Legendre transformation:

$$g(H, T) = f(h, T) - \frac{1}{4\pi}hH. \quad (2)$$

The second term in this equation is the microscopic version of $(4\pi)^{-1}BH$, since the magnetic induction $B(\mathbf{r}) = \langle h(\mathbf{r}) \rangle$.

The critical magnetic field H_c is defined as the field which establishes the following condition on the Gibbs energies in the normal and superconductor regions:

$$g_n(H_c, T) = g_s(H_c, T). \quad (3)$$

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[1]For a more complete discussion on conventional SC, see [5] or [6]

We also have the following relations, the first one due to the Meissner effect:

$$g_s(H,T) = f_s(T),$$

$$g_n(H,T) = f_n - \frac{H^2}{4\pi}.$$

From Eq. (1), we can obtain a correspondence with the parameters of the theory:

$$f_n(T) - f_s(T) = \frac{a_1^4}{4b}.$$

One finds:

$$\frac{H_c^2}{4\pi} = \frac{a_1^4}{2b} = \frac{\Phi_0^2}{32\pi^3\lambda^2\xi^2}, \tag{4}$$

where $\Phi_0 = hc/e^*$ is the quantum of magnetic flux, $\lambda = (m^*c^2/4\pi e^{*2}v^2)^{1/2}$ is the penetration depth and $\xi = (\hbar^2/m^*a_1^2)^{1/2}$ is the coherence length.

There is also another critical field, H_{c2}, which is the strongest field allowed in a superconducting region before the system goes to the normal state. It is given by:

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2} = \sqrt{2}\kappa H_c,$$

where, as usual, $\kappa = \lambda/\xi$. We see that if κ is equal to its critical value $1/\sqrt{2}$, $H_c = H_{c2}$ (more on this in Sections 2.1 and 3.2).

2.1 Surface Energy

As is well known, a superconductor will behave in one of two ways when placed in a magnetic field. In a type I SC, macroscopic normal regions where $h = H_c$ will form, while in a type II SC, a lattice of vortices of flux Φ_0 will appear. One way to determine under which circumstances a SC is of type I or II is by studying the energy per unit area of a boundary between a normal region and a SC region. If this energy is positive, the system will tend to minimize the interface's surface area in order to lower this energy as much as possible, indicative of a type I SC. Similarly for negative surface energy, this surface area will be maximized, inducing type II behavior.

Analytically, the surface energy is given by:

$$\hat{G}_s = \frac{\Phi_0^2\kappa^2}{32\pi^3\lambda^3}I = \frac{H_c^2\lambda}{4\pi}I,$$

where:

$$I = \int_{-\infty}^{\infty} ds \left\{ \frac{1}{2}(1 - f^4) + h^2 - \sqrt{2}h \right\}. \tag{5}$$

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Consider therefore a field configuration between the normal state for $x \to -\infty$ and the SC state for $x \to +\infty$:

$$\phi = \phi(x), \quad \mathbf{A} = A(x)\hat{y}.$$

Let us go to a description in terms of dimensionless variables, setting:

$$x = \lambda s, \quad \phi = vf, \quad A = \frac{c\hbar}{e^*\xi}a, \quad \hat{h} = \frac{c\hbar}{e^*\xi\lambda}h. \tag{6}$$

We obtain the following dimensionless free energy:

$$F = \int ds \left\{ f_n + \frac{1}{\kappa^2}f'^2 + (a^2 + f^2 - 1)f^2 + h^2 \right\}. \tag{7}$$

Minimizing F yields the following equations (the so-called 1D GL equations):

$$\frac{1}{\kappa^2}\frac{d^2f}{ds^2} + (1 - a^2)f - f^3 = 0, \tag{8}$$

$$\frac{d^2a}{ds^2} - af^2 = 0. \tag{9}$$

The sign of the surface energy can be determined qualitatively in the limit of either small or large value of κ in Eq. (8), by looking at the length scales of h and f. By construction (see Eq. (6)), the length scale of h is $l_h = 1$, while that of f turns out to be $l_f = 1/(\sqrt{2}\kappa)$.
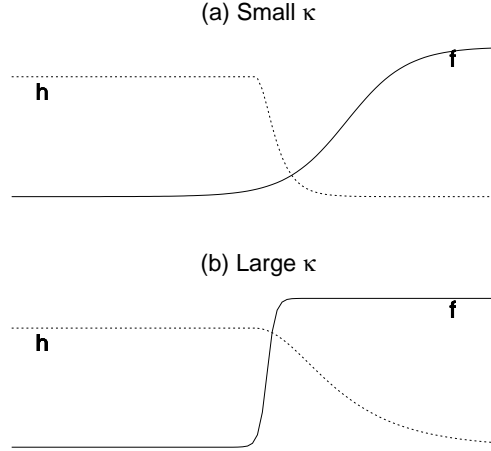

Figure 1: Field solution behaviors for the conventional superconductor (a) type I (b) type II

From Fig. 1, we can see how the sign of the surface energy is determined by the behavior of f and h in those limiting cases. In the first limiting case

(Fig. 1a) which corresponds to a type I sample ($\kappa \ll 1$, so that $l_f \gg l_h$), the energy comes from the expulsion of the magnetic field within the SC region. More explicitly, h is rapidly expulsed at the interface while f slowly assumes its asymptotic value. This gives rise to a positive scalar field energy density without a cancelling negative magnetic energy; the surface energy is positive. Fig. 1b shows the reverse case of a type II sample, where f quickly reaches its asymptotic behavior while the magnetic field slowly decreases, resulting in a negative surface energy.

The critical value of κ is the one for which the surface energy (or equivalently, the integral (5)) is zero. The well-known result $\kappa_c = 1/\sqrt{2}$ can be determined either numerically [2], or by making the oracular observation that when f and h obey the equations of motion for $\kappa = 1/\sqrt{2}$, then the integrand of (5) is zero[3]. This observation can be explained by a recasting of the work of Bogomol'nyi [8] (to be discussed in the next section) into a form appropriate to the surface energy.

2.2 Vortex Energetics

An alternative approach to determining the behavior of a superconductor when placed in a magnetic field is the study of the energy of vortices as a function of their winding number m. In this section, we will see what information can be drawn from the analytic expression, as well as an important limiting case, $m \to \infty$. In this limit, we will also show how one can establish a link between the vortex and surface energies.

As mentioned above, when placed in a magnetic field, the system will eventually find the configuration which minimizes the free energy. We can construct a function $\mathcal{F}(m) = F(m)/m$ which is the free energy of a vortex of winding number m divided by the winding number, or in other words the free energy per flux quantum of an m-vortex. For large m, if $\mathcal{F}(m)$ is an increasing function of m, it is more energetically expensive for a magnetic field to penetrate in vortices of large winding number; rather, a lattice of vortices of winding number 1 will form. This, clearly, is what we expect of a type II superconductor. Similarly, if $\mathcal{F}(m)$ is a decreasing function of m, the least energetic configuration will be a macroscopic normal region containing a large number of flux quanta; this is what we expect of a type I superconductor.

For the m-vortex configuration, we use an ansatz that is generalized to winding number m:

$$\phi = f(r)e^{im\theta}, \quad \hat{A}_i = \frac{a_1 c \sqrt{m^*}}{e^*} \epsilon_{ij} \frac{r_j}{r} a(r).$$

Again we can use the dimensionless variables defined in (6). Here, however, s denotes the radial variable in a 2D plane, as opposed to the cartesian coordinate

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[2] A numerical calculation of the surface energy is given in [7].

[3]Note that there is no need for the integrand of (5) itself to be zero; the weaker condition that the integral is zero would suffice. Indeed, we will see in the next section that in the SO(5) model the analogous integrand is *not* zero when the integrated surface energy is zero.

orthogonal to the domain wall.

Using these changes in Eq. (1) yields the vortex energy:

$$F_v = \int_0^\infty ds \frac{s}{2} \left\{ \frac{1}{2} + \left(a' + \frac{a}{s} \right)^2 + \frac{1}{\kappa^2} \left(f'^2 + \left(\frac{m}{s} + \kappa a \right)^2 f^2 \right) \right.$$
$$\left. - (1 - f^2) f^2 \right\}, \qquad (10)$$

as well as the equations of motion:

$$\frac{1}{\kappa^2} \left(f'' + \frac{1}{s} f' - \left(\frac{m}{s} + \kappa a \right)^2 f \right) + f - f^3 = 0, \qquad (11)$$

$$h' + \left(\frac{m}{\kappa s} + a \right) f^2 = 0. \qquad (12)$$

It is possible to rewrite Eq. (10) in the following form [8]:

$$2\kappa^2 F(m) = \int_0^\infty s\,ds \left\{ \left[f' - \left(\frac{m}{s} + \kappa a \right) f \right]^2 \right.$$
$$\left. + \left[\kappa \left(a' + \frac{a}{s} \right) + \frac{1}{2}(1 - f^2) \right]^2 + \frac{1}{2}(\kappa^2 - \frac{1}{2})(1 - f^2)^2 + \frac{d\chi}{ds} + \frac{1}{s}\chi \right\}, \qquad (13)$$

where $\chi = \kappa a(f^2 - 1) + mf^2/s$. Integrating the last two terms yields the winding number m, independent of the details of the field. Clearly, if $\kappa = 1/\sqrt{2}$, the third term vanishes and F consists of the winding number m plus two positive semidefinite terms; it will be minimized if they are zero. One can see that these terms are in fact zero if f and a satisfy the equations of motion. In this case, $\mathcal{F}(m) = F(m)/m = 1$ independent of m, which indeed corresponds to critical behavior with regard to vortice lattice stability, as there is no preference for either type I (coalescing) or type II (lattice) behavior.

We can determine the behavior of $\mathcal{F}(m)$ numerically for a large value of the winding number m. Our numerical results already suggest that $\mathcal{F}(m)$ reaches asymptotically a constant value at $m \to \infty$. In this limiting case, the core of the vortex is very wide, and the fields fluctuate rapidly at only one spatial position– that is at the vortex perimeter. Let us define this position as s_m. Analytically, we will see this more clearly by inserting the limiting forms of the field into Eq. (13), which are:

$$s < s_m: \qquad f \to 0, \qquad a \to -\frac{H_c}{2} s, \qquad h = 0,$$

$$s > s_m: \qquad f \to 1, \qquad a \to -\frac{m}{\kappa s}, \qquad h = -a' - \frac{a}{s} \to \frac{1}{\sqrt{2}}.$$

After integration, the second and third terms of Eq. (13) become, respectively:

$$\left(\frac{1}{2} - \kappa H_c \right)^2 \frac{s_m^2}{2} \qquad \text{and} \qquad \frac{1}{4}(\kappa^2 - \frac{1}{2})s_m^2.$$

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We have m and the remaining bracketed term, which does not contribute to the integral (this, of course, stems from our choice of step-functions as the role of fields f and h, since we essentially remove the presence of the surface altogether). Let us rewrite the variable s_m in terms of the quantum of magnetic flux Φ_0, using our $m \to \infty$ ansätze:

$$\Phi_0 = \frac{1}{m} \int_0^\infty h d^2 x = \frac{\pi s_m^2 H_c}{m}.$$

Therefore:

$$s_m^2 = \frac{2m}{\kappa H_c}.$$

We find:

$$\mathcal{F}(m) = \frac{1}{\sqrt{2}\kappa} \qquad\qquad (m \to \infty).$$

If we add thickness to the interface, the first term of Eq. (13), which was vanishing, will instead be proportional to $\sqrt{m}$ (i.e., a "perimeter" term $\propto s_m$).

We are now able to establish a correspondence between the free energies of the vortex and wall domain configurations. Indeed, for a large value of m, the perimeter of the vortex is large enough to significantly lower the curvature of the vortex locally, likening a given small region into the form of a domain wall. We should expect to see confirmation of this, using the appropriate approximations. We will do this here.

Let us start from the Gibbs energy of the vortex and derive an expression that clearly shows the relation with the domain wall Gibbs energy. We have:

$$\hat{G}_v = \hat{F}_v - \frac{H_c}{4\pi}\Phi_0 m = \frac{\Phi_0^2}{16\pi^2\lambda^2}\left(2\kappa^2 F(m) - \sqrt{2}\kappa m\right).$$

Substituting $F(m)$ with Eq. (13) and expanding the terms in square parentheses:

$$\hat{G}_v = \frac{\Phi_0^2}{16\pi^2\lambda^2}\left(m + \int_0^\infty s ds \left\{ \frac{\kappa^2}{2}(1-f^2)^2 + \kappa^2 h^2 - \kappa h(1-f^2) \right.\right.$$
$$\left.\left. + f'^2 - 2ff'\left(\frac{m}{s} + \kappa a\right) + \left(\frac{m}{s} + \kappa a\right)^2 f^2 \right\} - \sqrt{2}\kappa m \right). \qquad (14)$$

We notice that the following terms are equal to a familiar quantity:

$$-\kappa h(1-f^2) - 2ff'\left(\frac{m}{s} + \kappa a\right) = -\frac{d\chi}{ds},$$

which, once integrated, cancels the m term. If we take Eq. (11) and multiply it by $f\kappa^2$, we have:

$$\left(\frac{m}{s} + \kappa a\right)^2 f^2 = ff'' + \kappa^2(f^2 - f^4). \qquad (15)$$

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Next, we change variables $s \to z = s - s_m$, where it is understood that the region of interest is $|z| \ll |s_m|$. Therefore

$$\int_0^\infty s \, ds \to \int_{-\infty}^\infty (z + s_m) \, dz \approx \int_{-\infty}^\infty s_m \, dz.$$

Substituting (15) into (14) and simplifying, we have:

$$\hat{G}_v = \frac{\kappa^2 \Phi_0^2}{16\pi^2 \lambda^2} \left(\int_{-\infty}^\infty s_m dz \left\{ \frac{1}{2}(1 - f^4) + h^2 - \sqrt{2}h \right\} + \frac{s_m}{\kappa^2} \int_{-\infty}^\infty \frac{d(ff')}{dz} dz \right). \quad (16)$$

We recognize the first integral: it is s_m times I, as defined by Eq. (5); the second integral is zero due to the boundary conditions. Therefore, the link between the Gibbs energies between the vortex and domain wall case is summed up by the following relation, valid for large m:

$$\hat{G}_v = \frac{\kappa^2 \Phi_0^2 s_m}{16\pi^2 \lambda^2} I = \hat{G}_s(2\pi\lambda s_m).$$

In this section, we have studied vortex energetics as an alternative approach to the determination of the type (I vs. II) of a given superconductor, in the conventional case. We now turn our attention to the case of SO(5) superconductivity.

3   SO(5) Superconductivity

Having briefly reviewed the basics of conventional SC, let us turn to the case of SO(5) superconductivity. As mentioned in the introduction, the SO(5) model attempts to unify the superconducting and antiferromagnetic aspects of high-temperature superconductors. The order parameters for these two phenomena form a 5-dimensional real field whose dynamics has an approximate SO(5) symmetry, $(\phi_1, \phi_2, \eta_1, \eta_2, \eta_3)$, where $\phi = \phi_1 + i\phi_2$ and $\eta = (\eta_1, \eta_2, \eta_3)$ are, respectively, the order parameters of superconductivity and antiferromagnetism. This so-called superspin is described by a free energy which will be given below; explicit SO(5) breaking enables spontaneous breaking of either the SO(2) symmetry of ϕ or the SO(3) symmetry of η, describing SC and AF, respectively.

Mathematically, the low-energy effective theory is described in terms of the following free energy [4]:

$$\hat{F} = \int d\mathbf{x} \left(\frac{\hat{\mathbf{h}}^2}{8\pi} + \frac{\hbar^2}{2m^*} \left| \left(-i\nabla - \frac{e^*}{\hbar c} \hat{\mathbf{A}} \right) \phi \right|^2 + \frac{\hbar^2}{2m^*}(\nabla\eta)^2 + V(\phi, \eta) \right), \quad (17)$$

where $\hat{\mathbf{h}} = \nabla \times \hat{\mathbf{A}}$ is again the microscopic magnetic field.

The nature of the ground state can be determined by examining the potential. We consider the most general symmetry-breaking potential including even powers of the fields up to quartic terms,

$$V(\phi, \eta) = -\frac{a_1^2}{2}\phi^2 - \frac{a_2^2}{2}\eta^2 + \frac{b_1\phi^4 + 2b_3\phi^2\eta^2 + b_2\eta^4}{4},$$

where we have written $\phi = |\phi|$ and $\eta = |\eta|$. This potential is invariant under an $SO(3) \times SO(2)$ symmetry. $SO(5)$ symmetry would be attained by setting the two mass parameters and the three quartic coupling constants to be equal; for an approximate symmetry their values are approximately equal. To simplify the analysis we will set the quartic parameters to a common value $b_1 = b_2 = b_3 = b$; explicit symmetry breaking in the potential is found only in the quadratic terms.

Since we are interested in the magnetic properties of the superconducting state, we restrict ourselves to parameters which describe a SC ground state. This is done by requiring that the global minimum of V be on the ϕ axis. The ground state will then have the value $(\phi, \eta) = (v, 0)$, where $v \equiv a_1/\sqrt{b}$; the global condition is fulfilled if $\beta \equiv a_2^2/a_1^2 < 1$. Note that we recover $SO(5)$ symmetry if $\beta = 1$ and, therefore, this is the value corresponding to critical doping. For $\beta > 1$ the ground state is antiferromagnetic.

The goal of this section is to calculate the surface energy, whose sign indicates the type of superconductor described by the model. We will first set up the physical context of the surface energy and derive the critical fields H_c and H_{c2}. These enable us to determine the value of κ_c as a function of β. Subsequently, we will give numerical calculations which confirm our analytical results.

3.1 One Dimensional Free Energy

As outlined in Section 2.1, the surface energy is the Gibbs free energy per unit area of a domain wall separating a normal (non-superconducting) phase in the thermodynamic critical field H_c and a superconducting phase in the absence of a magnetic field. We use the same ansatz for ϕ and A as mentioned in the beginning of Section 2.1, and with the AF order parameter written $\eta = \eta(x)\hat{x}$, the free energy (17) becomes:

$$\hat{F} = \int dx \left\{ f_n + \frac{\hbar^2}{2m^*}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2m^*}\left(\frac{e^*}{c}\right)^2 A^2\phi^2 + \frac{1}{8\pi}\left(\frac{dA}{dx}\right)^2 \right.$$
$$\left. + \frac{\hbar^2}{2m^*}\left(\frac{d\eta}{dx}\right)^2 + V(\phi, \eta) \right\}. \qquad (18)$$

It will again be useful to use the dimensionless quantities defined in (6). Writing $\eta(x) = vn(s)$, we find, after some algebra, the following dimensionless free energy:

$$F = \int dx \left\{ f_n + \frac{1}{\kappa^2}(f'^2 + n'^2) + a^2f^2 + h^2 - f^2 - \beta n^2 \right.$$

$$+\frac{1}{2}(f^4 + 2f^2n^2 + n^4)\Bigg\}. \tag{19}$$

Minimizing with respect to f, n and a, we obtain the SO(5) GL equations:

$$\frac{1}{\kappa^2}\frac{d^2f}{ds^2} + \left(1 - a^2\right)f - f^3 - n^2f = 0, \tag{20}$$

$$\frac{d^2a}{ds^2} = af^2, \tag{21}$$

$$\frac{1}{\kappa^2}\frac{d^2n}{ds^2} + \beta n - n^3 - nf^2 = 0. \tag{22}$$

Note that if we set $n = 0$ in the first two equations, we recover the conventional GL Eqs. (8-9), as expected. Note also that we are now left with two parameters (κ, β) rather than three $(a_1^2, a_2^2$ and $b)$. The first of these two is the traditional GL parameter $\kappa = \lambda/\xi$, which is usually quite large for high T_c SC (typically between 15 and 100). The second fundamental parameter is β, which is related to the doping of the system away from critical doping (that corresponding to the AF/SC transition). β can be written as a function of the chemical potential μ as $\beta = 1 - 8m^*\xi^2\chi(\mu^2 - \mu_c^2)/\hbar^2$, where χ is the charge susceptibility and μ_c is the critical value of the chemical potential [2].

3.2 Calculation of H_c and H_{c2}

When a critical magnetic field is applied in a region, the SC order parameter will be forced to zero, destroying superconductivity. In order to minimize the Gibbs free energy, a nonzero AF order parameter will be induced. This is because, f being held to zero by the magnetic field, the potential in this region is minimized if $n = \sqrt{\beta}$, as can be seen from the potential terms in (19). The value of the critical field is determined by the competition between the positive potential energy of the AF state (relative to the SC ground state), on the one hand, and the negative magnetic energy $(-H_c^2/8\pi)$, on the other.

Let us compute, first of all, the critical magnetic field H_c. The result differs from the conventional one because an AF order parameter will now appear in the normal region.

Far into the superconducting region $(x \to \infty)$, we get the following behavior for the fields: $h = 0$, $f = 1$, $n = 0$. Thus from Eqs. (2) and (18),

$$g_s(H,T) \approx g(h = 0, T) = f_s(T) = f_n(T) - \frac{a_1^4}{4b}. \tag{23}$$

In the normal region $(x \to -\infty)$, we include a nonzero AF order parameter, and set $h = H$, $n = \sqrt{\beta}$, $f = 0$, giving,

$$g_n(H,T) = f_n(T) - \frac{a_2^4}{4b} - \frac{H^2}{8\pi}. \tag{24}$$

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By equating the two Gibbs free energies, we find the following β-dependent expression for the critical magnetic field:

$$H_c(\beta) = H_c^0 \sqrt{1 - \beta^2}, \tag{25}$$

where H_c^0 is the conventional expression (4). Note that $H_c(\beta)$ will refer to the critical field in the SO(5) model, and must be distinguished from the one in the GL theory (from now on, we will write H_c^0). Eq. (25) tells us that for $\beta = 1$, $H_c(\beta)$ vanishes. In fact, this value of β corresponds to the critical value, where the transition from SC ($\beta < 1$) to AF ($\beta > 1$) occurs; it is natural that any magnetic field, however small, will break SC, since SC is not energetically favored over AF: their free energies are equal. As β decreases, $H_c(\beta)$ reaches a maximum when $\beta = 0$, i.e., when the sample is at maximal doping. In this case, the value of $H_c(\beta)$ is the same as in conventional GL theory with the same κ. Thus, far in the normal state, the critical field will take the value $H_c(\beta) = (1/\sqrt{2})\sqrt{1 - \beta^2}$ since $H_c^0 = 1/\sqrt{2}$ in the conventional SO(2) case (see Section 2.2).

Also of interest is the critical field H_{c2}. For a type I SC, at $T < T_c$, when $H > H_c^0$, the SC phase is destroyed. After that, if H is reduced, the SC is recovered at $H_{c2} < H_c^0$ (phenomenon of supercooling). For a type II SC, H_{c2} corresponds to the maximum magnetic field in which a vortex lattice can appear; it is always higher than H_c^0. Then, H_{c2} is the strongest field which permits the SC phase to occur.

Near H_{c2}, the order parameter is small (the system is near the transition), and we use the linearized SO(5) GL equations. On the other hand, we can show that, at first order, the microscopic field $\mathbf{h}$ can be taken equal to the applied field $\mathbf{H}$. Beginning with the free energy Eq. (17), we choose the Landau gauge $\mathbf{H} = H\hat{z}$ and $A\hat{y} = Hx$. Linearizing around $\phi = 0$ and $\eta = a_2/\sqrt{b}$ (the values of fields at the transition), the equation for the SC order parameter takes the form [9]:

$$\left[\nabla^2 + \frac{4\pi i}{\Phi_0}\tilde{H}x\frac{\partial}{\partial y} - \left(\frac{2\pi\tilde{H}x}{\Phi_0} \right)^2 \right]\psi = -\kappa^2(1 - \beta)\psi,$$

where we have set $\tilde{H} = \lambda^2 H$, $\phi = v\psi$ and Φ_0 is the flux quantum.

This equation can be put in an harmonic oscillator equation form for x, by seting $\psi = f(x)\exp(-i(k_y y + k_z z))$ and the quantization of the energy leads to a set of allowed values of the magnetic field. The highest possible value of H_n, then, corresponds to the upper critical field and we find

$$H_{c2}(\beta) = \frac{\Phi_0}{2\pi\xi^2}(1 - \beta) = \sqrt{2}\kappa H_c^0(1 - \beta). \tag{26}$$

Note that it is the conventional critical field that appears in the right hand of Eq. (26).

In the GL theory of conventional SC, we have seen that the boundary between type I and type II SC occurs when $\kappa = \kappa_c = 1/\sqrt{2}$; at this value,

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$H_{c2} = H_c^0$. We can already anticipate the value of κ_c in the SO(5) model as the value for which these two critical fields are equal.

From (25) and (26) we find:

$$\kappa_c(\beta) = (1/\sqrt{2})\sqrt{(1+\beta)/(1-\beta)}. \tag{27}$$

Thus we are able to obtain an analytical expression for κ_c as a function of the doping. Two facts emerge from Eq. (27): firstly, we recover (as we must) the conventional result when $\beta = 0$, and secondly, for any β, a value of κ_c exists which separates type I and type II behavior (Figure 2).

As β approaches 1 (the SO(5) symmetric case), the instability of a vortex lattice (in the type I region) will be amplified, and therefore, to reach the type II region, we will need a higher κ. In practice, it is easier to vary the doping β (or μ) than the GL parameter κ in a sample.

We can reverse this point of view: if we consider a SC at a fixed value of κ, we will find that, for $\beta > \beta_c$ (defined implicitly in (27)), the sample will exhibit type I behavior. Thus, for high values of κ (i.e., $\kappa \gg 1$), we also need high β (i.e., β approaching 1) in order to see type I behavior. We will see in the next section that the numerical calculation of the surface energy agrees perfectly with these results.
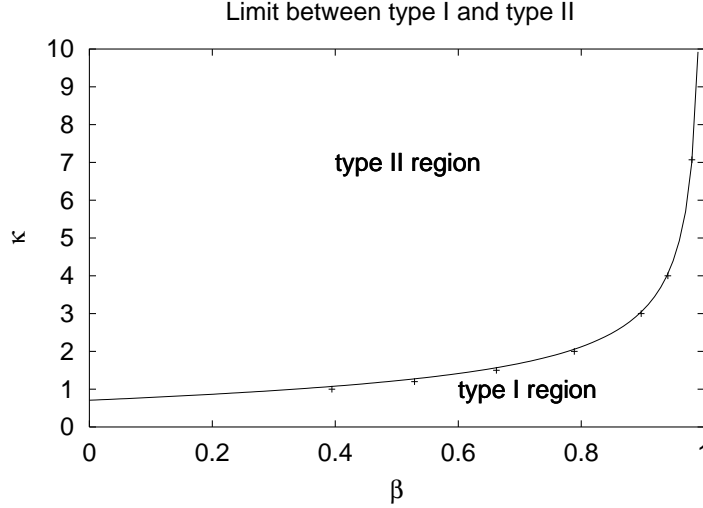

Figure 2: κ_c as a function of the doping β. The curve delimits the different types of SC. The points come from numerical calculations (see the end of Section 3.3 and Table 1).

3.3 Surface Energy

In this section, we will derive an expression for the surface energy in the SO(5) model, and rederive from it numerically the function $\kappa_c(\beta)$. We will begin by

analyzing qualitatively the surface energy; we will see that for κ sufficiently large (small), the surface energy is negative (positive). However, even this qualitative argument indicates that the critical value of κ depends on β, as anticipated in the previous section. We will then present numerical results since, as in the conventional model, the surface energy cannot be evaluated analytically; these numerical results agree with the conclusion of the previous section based on the critical fields.

To analyze qualitatively the surface energy, consider Fig. 3. In the normal region as $x \to -\infty$, the positive contribution to the energy coming from the scalar fields f and n exactly cancels the negative condensation energy of the magnetic field. In the SC region, as $x \to \infty$, the scalar field and magnetic energy densities individually are zero. Inside the surface, variations in the fields contribute to the surface energy. The details of these variations will determine the sign of the surface energy.

We can characterize the fields' variations by their characteristic lengths. From Eqs. (20-22), these are seen to be

$$l_h = 1, \quad l_f = \frac{1}{\sqrt{2}\kappa}, \quad l_n = \frac{1}{\kappa\sqrt{1-\beta}}. \tag{28}$$

At issue is whether in going from one asymptotic regime to the other, the scalar fields or the magnetic field are capable of changing more quickly. If the scalar fields vary faster, the magnetic field will penetrate into the superconducting region, and the negative magnetic energy density in this transition zone will not be accompanied by a positive scalar energy density, resulting in a negative surface energy. If the magnetic field varies faster, the situation is reversed and the surface energy will be positive. Thus we need to compare the larger of l_n and l_f with l_h. We see from Eq. (28) that l_n is always greater than l_f, so l_n determines how quickly the scalar fields can vary. Two cases are important to us when $\kappa \gg 1/\sqrt{2}$. First, when β is small (Fig. 3a), we find from (28) that $l_h > l_n$, and as described above, the surface energy will be negative. This corresponds to the behavior of a type II SC.

On the other hand, when β is large (Fig. 3b), we have the opposite situation: $l_h < l_n$, and the surface energy will be positive. This is the case of a type I SC.

By continuity, for any κ, there must exist a β where the surface energy is zero. This provides an alternative way of finding $\kappa(\beta)$, and must agree with Eq. (27).

We define the SO(5) surface energy in the same way as in Section 2.1. Subtracting Eq. (24) from Eq. (2) we are led to

$$\hat{G}_s = \int_{-\infty}^{\infty} dx \left\{ f_s - \frac{\hat{h}H_c}{4\pi} - \left(f_n - \frac{a_2^4}{4b} - \frac{H_c^2}{8\pi} \right) \right\}. \tag{29}$$

Using (18) we find, after some algebra,
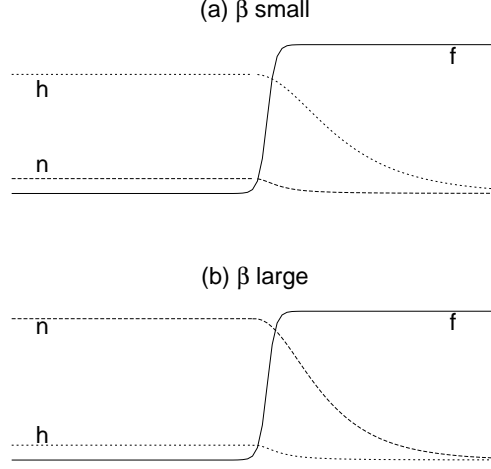
$$\hat{G}_s = \frac{(H_c^0)^2}{4\pi} \lambda I,$$

13

Figure 3: Field profiles in surface for $\kappa \gg 1/\sqrt{2}$. (a) β small (b) β large.

where

$$I = \int_{-\infty}^{\infty} ds \left\{ \frac{1}{2}(1 - f^4 - n^4) - f^2 n^2 + h^2 - \sqrt{2}h\sqrt{1 - \beta^2} \right\}, \qquad (30)$$

with λ and H_c^0 defined in Section 2. Therefore, the β dependence appears only in I, and the expression reduces to the conventional one, Eq. (5), when we set $n = \beta = 0$.

Of interest are the conditions under which this integral is zero. It will be recalled that in the SO(2) case, when $\kappa = 1/\sqrt{2}$, not only is the integral zero but the integrand is zero – a stronger statement, and one which is explained by Bogomol'nyi [8]. One might hope that the same feature would be found in the SO(5) model, along with an analogous analytical explanation. Unfortunately, such is not the case, and the surface energy must be studied numerically to find $\kappa_c(\beta)$.

We have solved Eqs. (20-22) numerically, using a relaxation algorithm. The boundary conditions appropriate for the problem are:

$$\begin{array}{llll} x \to -\infty : & f \to 0, & h \to \sqrt{1 - \beta^2}/\sqrt{2}, & n \to \sqrt{\beta}; \\ x \to \infty : & f \to 1, & h \to 0, & n \to 0. \end{array} \qquad (31)$$

Having found the solution, it is easy to compute the dimensionless surface energy I as a function of the doping β by numerical integration of (30). We present, in Fig. 4, the results for $\kappa = 0.707$ and $\kappa = 7.07$ respectively. The case $\kappa < 1/\sqrt{2}$ is of no interest since the surface energy is positive for all β.

Fig. 4a corresponds to κ_c for conventional SC. First of all, we see that the surface energy is zero for $\beta = 0$, as expected since, at this particular value, we

recover the SO(2) case. For all other values of β, we have a positive surface energy. Thus, the sample remains in the type I region for any value of the doping. This is in agreement with Eq. (27), as seen in Fig. 2. Note also that the energy reaches a maximum and decreases until it reaches zero again at $\beta = 1$. At this point, $h = 0$ throughout the sample (see Eq. (31)), and the ground state is SO(5) symmetric. Hence, no broken symmetry appears and for topological reasons (since the surface energy can be treated like a soliton), the surface energy must be zero.
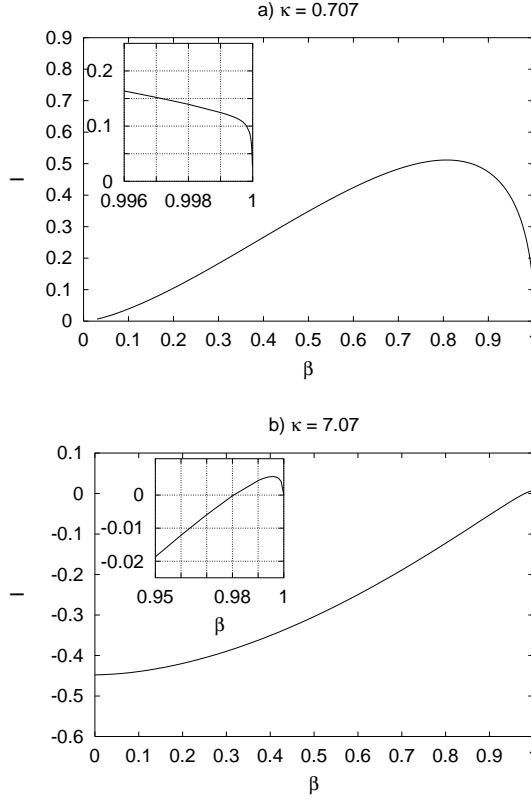

Figure 4: Surface energy $I(\beta)$ for a) $\kappa = 0.707$ and b) $\kappa = 7.07$.

A more interesting case is presented in Fig. 4b, corresponding to $\kappa = 7.07$ (in the conventional case, a type II superconductor). The curve starts at $\beta = 0$ at the conventional value and passes through zero at $\beta \approx 0.98$, in agreement with the analytical result (27). Below this critical value, the sample has type II behavior since the surface energy is negative. Above, the surface energy is positive, corresponding to type I behavior, i.e., vortices in this region are not energetically stable. Observe that the surface energy is again zero for $\beta = 1$.

We can determine in this way κ_c as a function of β; the results are reported in Table 1, and are in excellent agreement with Eq. (27).

β	$\kappa_c(\beta)$ (numerical)	From (27)
0.3945	1	1.07
0.5288	1.2	1.27
0.6620	1.5	1.57
0.7887	2	2.06
0.8976	3	3.04
0.9410	4	4.06
0.98	7.07	7.04
0.9998	70.7	70.71

Table 1: Comparison of κ_c obtained numerically via vortex energetics and surface energy, and analytically via Eq. (27).

3.4 Vortex Energetics in the SO(5) Model

As discussed in Section 2.2, the magnetic behavior of a superconductor can also be determined by studying vortex energetics [4]. Thus, one can calculate the free energy per quantum of magnetic flux of an m-vortex $\mathcal{F}(m) = F(m)/m$; if this function is of positive or negative slope, the superconductor is of type II or I, respectively; zero slope indicates the critical case, which can be expressed in terms of $\kappa_c(\beta)$. In the SO(5) model, the numerical calculation of $\mathcal{F}(m)$ for $\kappa = 7.07$ and 70.7 are given, respectively, in Fig. 2c and Fig. 2d in [4]. From the first of these, we can see that for $\beta < 0.98$ $\mathcal{F}(m)$ is of positive slope, while for $\beta > 0.98$ it is of negative slope; to a very good approximation, the slope is zero at $\beta = 0.98$. Thus, $\kappa_c(\beta = 0.98) = 7.07$. Similarly, Fig. 2d of the same reference indicates $\kappa_c(\beta = 0.9998) = 70.7$. A comparison of this method with the analytical method discussed in Section 3.2 is shown in Table 1; the agreement is excellent.

4 Summary and Conclusions

In this paper, magnetic properties of superconductors were discussed, with an eye towards analyzing the SO(5) model of high-temperature superconductivity. The Ginzburg-Landau theory of conventional superconductivity was discussed in Section 2, mainly to review some fundamental and well-known results and also to establish notation. The energy density of a surface separating superconducting and normal regions at critical applied field, and its importance for determining the behavior of the superconductor, were discussed. In addition, we reviewed the approach of Bogomol'nyi wherein the vortex energy is written in a form that shows clearly that at κ_c, the energy is proportional to the winding number of the vortex. Finally, we discussed an alternative approach to determining the critical field which is suggested by Bogomol'nyi's work, namely, studying the energy of an m-vortex as a function of its winding number. Both analytically and numerically, this approach can be shown to be equivalent to the surface energy approach.

In the second part, we introduced briefly the SO(5) model, and derived, in the context of that model, expressions for the critical magnetic fields, $H_c(\beta)$ and $H_{c2}(\beta)$. These enable us to calculate an analytic expression for $\kappa_c(\beta)$. Next, we studied the surface energy of a boundary between superconducting and non-superconducting regions in an applied critical field. Finally, we discussed briefly an alternative approach based on vortex energetics.

It is worth comparing the two numerical approaches to calculating κ_c, namely, by calculating the surface energy or by vortex energetics. In the former approach, one must solve the field equations (for instance, (20-22) in the SO(5) model), evaluate the surface integral (30), and vary β for fixed κ (or vice-versa) until the surface integral vanishes. In the latter approach, one must solve the vortex field equations (the generalization of (11) and (12) to SO(5)), evaluate the energy as a function of winding number, and determine for which κ, β the function $\mathcal{F}(m)$ is independent of m for large m. The latter method is somewhat more demanding numerically, essentially since several values of m must be considered for each κ, β, though it is perhaps more intuitive, since one can think of type I behavior as due to an energetic preference for the coalescence of many single vortices into one large m-vortex.

The main new result in this paper is Eq. (27), which expresses the value of κ as a function of β corresponding to the boundary between type I and type II superconductors. This result is rather important: it states that as $\beta \to 1$ (i.e., as the doping is reduced to its critical value), then even with a large value of κ, the material would be a type I superconductor, contrary to what one would naively expect. Some implications of this have been discussed in [4].

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